Contact process with sublattice symmetry breaking

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We study a contact process with creation at first- and second-neighbor sites and inhibition at first neighbors, in the form of an annihilation rate that increases with the number of occupied first neighbors. Mean-field theory predicts three phases: inactive (absorbing), active symmetric, and active asymmetric, the latter exhibiting distinct sublattice densities on a bipartite lattice. These phases are separated by continuous transitions; the phase diagram is re-entrant. Monte Carlo simulations in two dimensions verify these predictions qualitatively, except for a first-neighbor creation rate of zero. (In the latter case one of the phase transitions is discontinuous.) Our numerical results confirm that the symmetric-asymmetric transition belongs to the Ising universality class, and that the active-absorbing transition belongs to the directed percolation class, as expected from symmetry considerations.

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I. INTRODUCTION

An absorbing-state phase transition is a far-from-equilibrium transition between an active and an inactive phase. When a control parameter such as a creation or annihilation rate is varied, the system undergoes a phase transition from a fluctuating state to an absorbing one. Such transitions are currently of great interest, arising in a wide variety of problems, such as population dynamics, heterogeneous catalysis, interface growth, and epidemic spreading [1–5]. Interest on this class of phase transitions has been stimulated by recent experimental realizations [6,7].

As equilibrium statistical mechanics, absorbing-state phase transitions can be classified into universality classes. However, a complete classification of their critical behavior is still lacking; much numerical and theoretical work is focused on identifying the factors that determine the universality classes of these transitions.

The absorbing-state universality class associated with directed percolation (DP) has proven to be particularly robust. Directed-percolation-like behavior appears to be generic for absorbing-state transitions in models with short-range interactions and lacking a conserved density or symmetry beyond translational invariance [8]. By contrast, models possessing two absorbing states linked by particle-hole symmetry belong to the voter model universality class [9]. There are also models which, while free of absorbing states, cannot achieve thermal equilibrium because their transition rates violate the detailed balance principle. An example is the majority vote model. In its ordered phase a Z2 symmetry is spontaneously broken, leading to Ising-like behavior in two or more dimensions.

In this work we examine whether a spatially structured population model that suffers a phase transition to a single absorbing state can also exhibit a broken-symmetry phase. Our candidate for such a model is a modified contact process (CP) with suppression of activity at the nearest neighbors of active sites. We observe unequal sublattice populations for suitable choices of the control parameters.

The balance of this paper is organized as follows. In the next section we define the model and analyze its mean-field theory. In Sec. III we present our simulation results; Sec. IV is devoted to discussion and conclusions.

II. MODEL AND MEAN-FIELD THEORY

The CP [10] is a stochastic interacting particle system defined on a lattice, with each site i either occupied by a particle [σi(t) = 1] or vacant [σi(t) = 0]. Transitions from σi = 1 to σi = 0 occur at a rate of unity, independent of the neighboring sites. The reverse transition, at a vacant site, is only possible if at least one of its neighbors is occupied: The transition from σi = 0 to σi = 1 occurs at rate λr, where r is the fraction of nearest neighbors of site i that are occupied. In the absence of spontaneous creation of particles, the state σi = 0 for all i is absorbing. It can be shown [10] that for a certain critical value λc, the system undergoes a phase transition between the active and the absorbing state. Although no exact results are available, the CP has been studied intensively via series expansion and Monte Carlo simulation, and its critical properties are known to high precision [1–5].

In order to generate distinct sublattice occupations, we modify the basic contact process in two respects:

(i) In addition to creation at first neighbors, at rate λ1, we allow creation at second neighbors, at rate λ2. For bipartite lattices such as the square or simple cubic lattice, λ1 is the rate of creation in the opposite sublattice, while λ2 is the rate in the same sublattice as the replicating particle. Unequal sublattice occupancies are thus favored for λ2 > λ1.

(ii) We include a nearest-neighbor inhibition effect in the annihilation process. In addition to the intrinsic annihilation rate of unity, there is a contribution of μnσi, where n1 is the number of occupied first neighbors. Thus, if the occupation fraction ρA of sublattice A is much larger than that of sublattice B, any particles created in sublattice B will die out quickly, stabilizing the unequal sublattice occupancies.

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The one-site mean-field theory (MFT) for this model on a lattice of coordination number \( q \) is given by the coupled equations,

\[
\frac{d\rho_A}{dt} = -(1 + \mu q^2 \rho_A^2)\rho_A + (\lambda_1 \rho_B + \lambda_2 \rho_A)(1 - \rho_A) \quad (1)
\]

and

\[
\frac{d\rho_B}{dt} = -(1 + \mu q^2 \rho_B^2)\rho_B + (\lambda_1 \rho_A + \lambda_2 \rho_B)(1 - \rho_B), \quad (2)
\]

which are seen to be symmetric under \( \rho_A \leftrightarrow \rho_B \). Defining \( \rho = \rho_A + \rho_B \) and \( \phi = \rho_A - \rho_B \), these equations may be recast as

\[
\frac{d\rho}{dt} = (\Lambda - 1)\rho - \frac{\Lambda}{2} \rho^2 - \frac{\Delta}{2} \phi^2 - \frac{1}{4} \mu q^2 (\rho^2 - \phi^2) \rho \quad (3)
\]

and

\[
\frac{d\phi}{dt} = \left[ \Delta - 1 - \lambda_2 \rho - \frac{1}{4} \mu q^2 (\rho^2 - \phi^2) \right] \phi, \quad (4)
\]

where \( \Lambda \equiv \lambda_1 + \lambda_2 \) and \( \Delta \equiv \lambda_2 - \lambda_1 \).

From Eq. (3) it is evident that at this level of approximation, the extinction-survival transition occurs at \( \Lambda = 1 \), as one would expect. This transition leads to the symmetric active phase, characterized by \( \rho > 0 \) and \( \phi = 0 \). This phase is linearly stable for small \( \Delta \). As \( \lambda_2 \) is increased, with \( \lambda_1 \) constant, the symmetric active phase can become unstable, leading to a phase with \( |\phi| > 0 \). On further increasing \( \lambda_2 \), however, \( \rho \) can increase to the point that the symmetric active phase is again stable. (Perfect sublattice ordering corresponds to \( \rho = \phi \), which is only possible for \( \rho \leq 1 \); total densities larger than unity tend to suppress sublattice ordering.) Thus the asymmetric active phase should exist for some intermediate range of \( \lambda_2 \) values.

In the symmetric active phase, the stationary activity density is

\[
\rho = \frac{1}{2\kappa} \left[ \sqrt{(\Lambda/2)^2 + 4\kappa(\Lambda - 1)} - (\Lambda/2) \right] \quad (5)
\]

with \( \kappa \equiv \mu q^2/4 \). From Eq. (4), this solution is stable with respect to \( \phi \neq 0 \), when

\[
a_\phi \equiv \Delta - 1 - \lambda_2 \rho + \kappa \rho^2 < 0. \quad (6)
\]

The transition to the asymmetric phase occurs when \( a_\phi = 0 \), which implies

\[
(2\lambda_2 + \Lambda)^2(\Lambda - 1) = 2(\lambda_2 - 1)[8\kappa(\lambda_2 - 1) + (2\lambda_2 + \Lambda)\Lambda]. \quad (7)
\]

Equation (4) may be written as

\[
\frac{d\phi}{dt} = a_\phi \phi - \kappa \phi^3,
\]

showing the expected symmetry under \( \phi \rightarrow -\phi \), which in turn suggests that the transition between symmetric- and asymmetric-active phases is Ising-like. The two transitions (active/absorbing and symmetric/asymmetric) coincide at the point \( \lambda_1 = 0, \lambda_2 = 1 \). Figure 1 shows the phase diagram for \( \mu = 2 \), obtained using Eq. (7). Of particular note are the observations that (a) the asymmetric-active phase is observed only for intermediate values of \( \lambda_2 \), i.e., the phase diagram is reentrant, and (b) the asymmetric-active phase does not exist for \( \lambda_1 \) above a certain value, \( \lambda_1^*(\mu) \). For \( \mu = 2 \), for example, \( \lambda_1^* = 7.1443 \). Also shown in Fig. 1 are simulation results (details of our simulation method are discussed in the following section). MFT and simulation are in qualitative agreement, but as is usually the case, MFT overestimates the regions in parameter space corresponding to the active and ordered phases. In particular, MFT overestimates \( \lambda_1^* \) by about an order magnitude.

The line \( \lambda_1 = 0 \) is special, since the subspaces with \( \rho_A > 0 \) and \( \rho_B = 0 \) (and vice versa) are also absorbing. MFT yields the following phases: absorbing for \( \lambda_2 < 1 \); asymmetric-active-I for \( 1 < \lambda_2 < \lambda_1^*(\mu) \); asymmetric-active-II for \( \lambda_1^*(\mu) < \lambda_2 < \lambda_1^{**}(\mu) \); and symmetric-active for \( \lambda_2 > \lambda_1^{**} \). The difference between phases asymmetric-active-I and -II is that in the former, \( \rho = \phi \) (i.e., only one of the sublattices is populated), whereas in the latter \( \rho > \phi > 0 \). The transitions between these phases are continuous. (Naturally, the asymmetric-active-II and symmetric-active phases can be reached only from initial conditions with both sublattices populated.) For \( \mu = 2 \), for example, we find \( \lambda_1^* = 30.97 \) and \( \lambda_1^{**} = 41.65 \).

We also studied a version in which the suppression (enhanced annihilation rate) is a linear function of the number of occupied first neighbors, i.e., the mean-field equations

\[
\frac{d\rho_A}{dt} = -(1 + \mu q \rho_B)\rho_A + (\lambda_1 \rho_B + \lambda_2 \rho_A)(1 - \rho_A), \quad (8)
\]

and similarly for \( \rho_B \). The phase diagram is qualitatively similar to that found above, but the region occupied by the asymmetric-active phase is much reduced. For \( \mu = 2 \), for example, sublattice ordering is found only for \( \lambda_1 \leq 0.6805 \).

Since mean-field theory offers at best a qualitative description of the model, we turn, in the following section, to simulation.

### III. SIMULATIONS

We performed extensive Monte Carlo simulations of the model on square lattices of linear size \( L = 20, 40, \ldots, 320 \) sites, with periodic boundaries. The simulation algorithm is as follows. First, a site is selected at random. If the site is occupied, it creates a particle at one of its first neighbors with a probability \( p_1 = \lambda_1/(1 + \lambda_1 + \lambda_2 + \mu n^2) \) or at one of its
second neighbors with a probability $p_2 = \lambda_2/(1 + \lambda_1 + \lambda_2 + \mu n_2^2)$. With a complementary probability $1 - (p_1 + p_2)$ the site is vacated. (In order to improve efficiency the sites are chosen from a list which contains the currently $N_{occ}$ occupied sites; we increment the time by $\Delta t = 1/N_{occ}$ after each event.) For simulations in the subcritical and critical absorbing regime, we employ the quasistationary (QS) simulation method detailed in Ref. [11] to further improve efficiency.

In a series of studies using $\mu = 2.0$, we verify that for low values of $\lambda_1$ and $\lambda_2$ the system becomes trapped in the absorbing (inactive) state. When we increase $\lambda_1$ and/or $\lambda_2$ the system undergoes a phase transition from the inactive to the active symmetric (AS) phase. For example, for $\lambda_1 = 0.2$, the absorbing transition occurs at $\lambda_2 = 1.5620(5)$. (Note that at this point $\lambda_1 + \lambda_2$ is somewhat greater than the critical value for the basic process, $\lambda_c = 1.6488(1)$, on the square lattice [12].) At the critical point, the quasi-stationary order parameter is expected to decay as a power law, $\rho \sim L^{-\beta/\nu}$.

In Fig. 2, we show that our data follow a power law, $\tau \sim L^{-\beta/\nu}$. This value again is in very good agreement with the best known DP value $\nu = 0.79(1)$, in agreement with the DP value $\nu = 0.797(3)$ [13]. The lifetime of the QS state $\tau \sim L^{\ln \rho/\nu}$ at criticality, where we find $m = 1.767(6)$. The moment ratio $m = (\rho^2)/(\rho)^2$ (see Fig. 3) is analogous to Binder’s fourth cumulant [14] at an equilibrium critical point in the sense that $m$ assumes a universal value $m_c$ at the critical point.

In the active phase, we find that as we increase $\lambda_2$ at fixed $\lambda_1$, there is a transition to a state with unequal particle densities in the two sublattices ($\phi \neq 0$), that is, to the active asymmetric (AA) phase. A typical configuration in the AA phase is shown in Fig. 4. Our observation of a bimodal probability distribution for the order parameter $\phi$ (see Fig. 5) confirms the existence of $\phi$.
the AA phase. Although MFT predicts a symmetry-breaking transition in any number of dimensions, we do not expect such a phase transition in one dimension; simulations on a ring confirm this.

In order to classify the AA-AS transition, we perform a finite-size scaling study. First, we investigate the behavior of the reduced Binder cumulant \[ U_4 = 1 - \frac{\langle \phi^4 \rangle}{\langle \phi^2 \rangle^2} \] (9) The intersection points of the cumulants for successive pairs of sizes depend rather weakly on the sizes, providing a good estimate for the critical point. The value of the cumulant at the critical point approaches a universal value as \( L \to \infty \). In Fig. 6 we show the results for the case \( \mu = 2.0 \) and \( \lambda_1 = 0.2 \). The curves for different sizes intersect at \( \lambda_2 = 3.940(5) \) and again at \( \lambda_2 = 31.92(6) \) when \( L \to \infty \).

Extrapolating to infinite size, we estimate \( U_{4,c} = 0.605(10) \) at the transitions, consistent with the \( U_{4,c} = 0.61069 \ldots \) [16] for the two-dimensional Ising model with fully periodic boundary conditions. For values of \( \lambda_2 \) between the transitions points, the cumulant approaches \( 2/3 \), as expected in an ordered phase that breaks a \( Z_2 \) (i.e., up-down) symmetry. Further evidence for Ising-like behavior is furnished by the order-parameter variance. At the critical point, \( \text{var}(\phi) = \langle \phi^2 \rangle - \langle \phi \rangle^2 \) is expected to scale so: \( L^2 \text{var}(\phi) \propto L^{\gamma/\nu} \); this is confirmed in Fig. 7. A fit to the data yields the exponent ratio \( \gamma/\nu = 1.76(5) \); again, in good agreement with the known value of \( \gamma/\nu = 7/4 \) for the Ising model in two dimensions.

For the \( \lambda_1 = 0 \), simulations show no evidence of the asymmetric-active-II phase predicted by MFT. Simulations reveal only two phase transitions along this line. The first, from the absorbing to the asymmetric active phase, occurs,
as expected, at the critical value of the basic CP on the square lattice. We find $\lambda_{2,c} = 1.6488(3)$ and confirm that this transition belongs to the DP class [see Fig. 8, where we find $m = 1.320(8), \beta/\nu_\perp = 0.81(1)$, and $\nu_0/\nu_\perp = 1.74(3)$, consistent with the values for DP in $2 + 1$ dimensions, cited above]. The second transition, between the active-asymmetric and the active-symmetric phases, is discontinuous. Figure 9 illustrates such transitions for $\mu = 0.2$ and $\mu = 2$. These results signal a qualitative failure of mean-field theory along the line $\lambda_1 = 0$.

IV. CONCLUSIONS

A contact process with creation at both nearest and next-nearest neighbors and suppression (in the form of an increased annihilation rate) at neighbors of active sites is shown to exhibit a broken-symmetry phase with sublattice ordering. The existence of a continuous symmetry-breaking transition is predicted by mean-field theory and confirmed in simulations on the square lattice. The asymmetric active phase exists for a restricted range of the second-neighbor creation rate, $\lambda_2$, when at first neighbors, $\lambda_1$, is sufficiently small. For a given value of the suppression parameter $\mu$, there is a certain value, $\lambda_1^*$, above which no sublattice order occurs. We note that the possibility of breaking $Z_2$ symmetry depends on the fact that creation rates $\lambda_1$ and $\lambda_2$ apply to different sublattices. We would not expect to observe this symmetry breaking if, for example, there were a creation rate $\lambda_1$ for both nearest- and second neighbors and a different creation rate at third-neighbor sites.

For $0 < \lambda_1 < \lambda_1^*$, the sequence of phases observed on increasing $\lambda_2$ from small values, while holding $\mu$ and $\lambda_1$ fixed, is as follows: absorbing, symmetric-active, asymmetric-active, and symmetric-active. The first transition belongs to the universality class of directed percolation, while the latter two are Ising like. For the special case of $\lambda_1 = 0$, the first transition is from absorbing to asymmetric-active and the second is discontinuous. For both values of $\mu$ studied we observe continuous transitions for $\lambda_1 > 0$. While increasing $\lambda_1$ makes the transition smoother, we cannot exclude the possibility of a discontinuous transition when $\lambda_1 > 0$ for larger values of $\mu$. A complete characterization of the re-entrant transition for $\mu > 2.0$, including the location of the possible tricritical points separating the line of second-order transitions from that of first-order transitions, is left for future work.

The absorbing-active transition at $\lambda_1 = 0$ marks the meeting of two lines of critical points, one of DP transitions and the other of Ising transitions. The meeting of two such lines has been studied in the context of a nonequilibrium Potts model by Droz et al. [17] and of the generalized voter model (GVM) by Al Hammal et al. [18]. The former study also finds an absorbing state in the DP class, while in the latter, after the Ising and DP lines join, the absorbing transition belongs to the GVM class. We note that in these studies the models have but two (symmetric) absorbing states (i.e., all plus and all minus), whereas for $\lambda_1 = 0$, our model has two (symmetric) absorbing subspaces (activity restricted to one sublattice) and a third, completely inactive absorbing state. In terms of long-time behavior, the absorbing-active phase transition takes the system from the fully inactive state to one with activity restricted to one sublattice. Starting from all sites active, at the critical point $\lambda_1 = 0, \lambda_2 = \lambda_{c,CP}$, fluctuations drive local regions into states with activity in only one sublattice, as well as fully inactive regions. We expect that these domains then coarsen, so eventually there is activity only in one sublattice; from then onward the dynamics is that of the basic contact process. Thus, it is not surprising that we observe DP-like static behavior. It is nevertheless possible that the short-time critical behavior includes a regime dominated by domain dynamics in the GVM. It also seems possible that the discontinuous AA-AS transition observed for $\lambda_1 = 0$ belongs to the voter model class. We intend to explore these questions in a future work.

Possible extensions of this work include precise determination of static and dynamic critical properties at the symmetry-breaking transition, studies of the effect of diffusion and/or anisotropy (which may generate modulated phases), and studies of a model in continuous space. The latter appears to be of particular interest in the context of patterns arising in ecosystems or bacterial colonies. We note that the suppression of activity at the nearest neighbors of active sites resembles biological lateral inhibition, known to be important in the visual system of many animals [19]. Thus extensions of the present study might be useful in the study of patterned activity on the retina. Finally, if the inhibition rate were a decreasing function (linear or quadratic) of the number of occupied nearest neighbors (i.e., $\mu < 0$), one should observe symbiotic coexistence of activity in the two sublattices. The latter is reminiscent of the Janzen-Connell hypothesis for the maintenance of tree species biodiversity in tropical rainforests [20].

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